Vector optimization : Singularities, Regularizations *

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Abstract

We discuss three scalarizations of the multiobjective optimization from the point of view of the parametric optimization. We analize three important aspects:

i) What kind of singularities may appear in the different parametrizations

ii) Regularizations in the sense of Jongen, Jonker and Twilt, and in the sense of Kojima and Hirabayashi.

iii) The Mangasarian-Fromovitz Constraint Qualification for the first parametrization.

keywords: multiobjective optimization, parametric optimization, singularities, regularizations

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1 Introduction

We consider the following multiobjective optimization problem:

$$min \quad \{ \ f_1(x), \ldots, f_L(x) \mid x \in M \ \} \tag{1}$$

where M is defined by

$$M = \{ \ x \in I\!R^n \mid g_i(x) = 0, \ i \in I, \ g_k(x) \leq 0, \ k \in K \ \} \tag{2}$$

and where $f_j, \ j = 1 \ldots L, \ g_k, \ k \in I \cup K$ are given functions , $I = \{1 \ldots m\}$ and $K = \{1 \ldots p\} \setminus I$, $p \geq m$.

We use the following well-known notions of "optimality" for a multiobjective optimization problem.

Definition 1.1 *(c.f. e.g. [3])*
A point $x \in I\!R^n$ is called an efficient point if

$$(\ f(x) + D \ ) \cap f(M) = \emptyset \tag{3}$$

where $f = (f_1, \ldots, f_L)$ and $D = -I\!R^L_+ \setminus \{0\}$.

The concepts of ϵ-efficient point and weakly efficient point are defined analogously writing

$$D_\epsilon = \{ \ y \in I\!R^L \setminus \{0\} \mid dist(y, D) \leq \epsilon \, ||y|| \ \}$$

and $D_d = int(D)$ respectively instead of D.

There are local versions of these concepts that have a trivial definition. We denote the set of all (loc) efficient points by $M_{eff}(M_{loceff})$ and analogously the sets $M^\epsilon_{eff}(M^\epsilon_{loceff})$ and M_{w-eff} $(M_{locw-eff})$.

It is posible to characterize these sets by means of the solution sets of certain parametric optimization problems (c.f. e.g. [4]). We use here the following parametrizations of the problem (1),(2).

1st Parametrization: (c.f. e.g. [4])

$$P_1(\mu) : \ min \ \left\{ \sum_{k=1}^{L} \lambda^0_k \, f_k(x) \mid x \in M, \ f_k(x) \leq \mu_k, \ k \in L \right\} \tag{4}$$

where $L = \{1, \ldots, L\}$, $\mu_k \in I\!R \cup \{+\infty\}$, $k \in L$ and $\lambda^0 \in -D_d$.

We denote by $\Psi_1(\mu)$, $\Psi_{1loc}(\mu)$, the set of global and local optimal points of the problem $P_1(\mu)$. The following relation is known. (c.f. e.g. [5])

$$M_{eff} = \bigcup_{\mu \in \mathbb{R}^L} \Psi_1(\mu), \quad M_{loceff} = \bigcup_{\mu \in \mathbb{R}^L} \Psi_{1loc}(\mu).$$

2nd.Parametrization: (c.f. e.g. [3, 4])

$$s(f(x),\mu) = \max_{\text{i} \in L} \lambda_i^0\,(\,f_i(x) - \mu_i\,) + \delta \sum_{k=1}^{L} \lambda_k^0\,(\,f_k(x) - \mu_k\,).$$

where $\delta \in (0,1)$ sufficiently small is fixed.

The problem

$$min \quad s(x,\mu), \quad x \in M \tag{5}$$

has the following properties: (c.f. e.g. [5])

$$\bigcup_{\mu \in \mathbb{R}^L} \Psi_2(\mu) \subset M_{eff}, \quad \bigcup_{\mu \in \mathbb{R}^L} \Psi_{2loc}(\mu) \subset M_{loceff}\ ,$$

$$M_{eff}^{\epsilon} \quad \subset \quad \bigcup_{\mu \in f(M_{eff}^{\epsilon})} s^{-1}(f(x),\mu) \cap \Psi_2(\mu) \quad \subset \quad \bigcup_{\mu \in \mathbb{R}^L} \Psi_2(\mu)$$

and

$$M_{loceff}^{\epsilon} \subset \bigcup_{\mu \in \mathbb{R}^L} \Psi_{2loc}(\mu).$$

Since s is not differentiable, we transform (5) into the equivalent problem (c.f. e.g. [3, 4])

$$P_2(\mu):$$
$$min \left\{ \delta \sum_{k=1}^{L} \lambda_k^0 (f_k(x) - \mu_k) + v \;\middle|\; \begin{array}{l} x \in M,\ \forall k \in L \\ \lambda_k^0\,(f_k(x) - \mu_k) \leq v \end{array} \right\} \tag{6}$$

3rd Parametrization: (c.f. e.g. [3, 4])

$$P_3(\mu):\ min\,\{\ v \mid x \in M,\ f_k(x) - v \leq \mu_k,\ k \in L\ \} \tag{7}$$

If we consider $\Psi_3(\mu)$ ($\Psi_{3loc}(\mu)$) as the projection of the global (local) solution set of (7) into the x space. We obtain the relations: (c.f. e.g. [5])

$$M_{w-eff} \quad = \quad \bigcup_{\mu \in \mathbb{R}^L} \Psi_4(\mu), \quad M_{locw-eff} \quad = \quad \bigcup_{\mu \in \mathbb{R}^L} \Psi_{4loc}(\mu).$$

We use the reduction of the multiparametric optimization problems (4), (6) and (7) to a sequence of one-parametric optimization problems presented in [4].These sequence can be generated by a dialogue procedure with the decision maker (c.f. e.g. [4, 3]).

We obtain from the dialogue procedure some $\mu^0,\ \mu^1 \in I\!R^L$ and we have to consider the following one-parametric optimization problems.

$$\begin{array}{c} P_i(t) = P_i(\ \mu(t)\ ), \quad i = 1, \ldots, 3, \\ \mu(t) = (1-t)\,\mu^0 + t\,\mu^1, \quad t \in [0,1] \end{array} \tag{8}$$

We denote by $M_i(t), i = 1, 2, 3.$ the corresponding feasible sets for the parametrizations (8).

From the dialogue procedure (c.f. e.g. [4]) we know that μ^1 expresses the wishes of the decision maker. He is mainly interested to know wherther his wish μ^1 was realistic or not. If the goal point μ^1 is a realistic one, then the decision maker wants to find a point $\tilde{x} \in M$ such that

$$f_j(\tilde{x}) \le \mu_j^1 \quad j = 1, \ldots, L$$

We call a point $\mu^1 \in I\!R^L$ a realistic goal if

$$M_1(\mu^1) = \{x \in M | f_j(x) \le \mu_j^1 \quad j = 1, \ldots, L\} \ne \emptyset$$

and a point $x \in M_1(\mu^1)$ a goal realizer.

Our purpose in these papper is to analyse the singularities and regularizations of these 3 parametrizations from the point of view of the parametric optimization theory.

2 Theoretical Background

We consider the following general one-parametric optimization problem:

$$P(t): \quad min\ \{\ f(x,t)\ |\ x \in M(t)\ \}$$

where

$$M(t) = \{\ x \in I\!R^n\ |\ g_i(x,t) = 0,\ i \in I,\ g_k(x,t) \le 0,\ k \in K\ \}$$

I and K are index sets defined as in (2). We first recall the class $\mathcal{F}$ of Jongen, Jonker and Twilt [8].

Definition 2.1 *(c.f. e.g. [3, 8])*
Σ_{gc} is the set of all (x,t) such that $x \in M(t)$ and the vectors

$$D_x f(x,t),\ D_x g_k(x,t),\ k \in I \cup K_0(x,t)$$

are linearly dependent, where $K_0(x,t) = \{\ j \in K \mid g_j(x,t) = 0\ \}$

Here $D_x f$ denotes for the row vectors of first partial derivatives with respect to x. If $(x,t) \in \Sigma_{gc}$ there exits numbers u_j, $j \in I \cup K \cup \{0\}$, such that:

$$u_0\, D_x f(x,t) + \sum_{j\, \in\, I \cup K} u_j\, D_x g_j(x,t) = 0 \tag{9}$$

where $u_j = 0$ for $j \in K \setminus K_0$.

Definition 2.2 *(c.f. e.g. [3])*

LICQ *The linear independence constraint qualification is satisfied at $\bar{x} \in M(\bar{t})$ if the vectors*

$$D_x g_j(\bar{x},\bar{t}),\ j \in I \cup K_0(\bar{x},\bar{t})$$

are linearly independent.

MFCQ *The Mangasarian Fromovitz constraint qualification is satisfied at $\bar{x} \in M(\bar{t})$ if:*

1. *$D_x g_i(\bar{x},\bar{t})$, $i \in I$ are l.i.*
2. *There exits a vector $\xi \in \mathbb{R}^n$ with:*
   - *$D_x g_i(\bar{x},\bar{t})\ \xi = 0,\ \ i \in I$*
   - *$D_x g_j(\bar{x},\bar{t})\ \xi < 0,\ \ j \in K_0(\bar{x},\bar{t})$*

If $\bar{x}$ is a local minimizer for $P(\bar{t})$ and LICQ or MFCQ is satisfied at $\bar{x}$, then $(\bar{x},\bar{t}) \in \Sigma_{gc}$ and we can find $\bar{u}$ in (9), such that $\bar{u}_0 = 1$ and $\bar{u}_j \geq 0$, $j \in K_0(\bar{x},\bar{t})$

Definition 2.3 *(c.f. [8])*

1. *We call every $(x,t) \in \Sigma_{gc}$ a generalized critical point.*
2. *If $(x,t) \in \Sigma_{gc}$ and in (9) we can find u, such that $u_0 = 1$ and $u_j \geq 0,\ j \in K_0(x,t)$, then we call (x,t) an stationary point.*

Define:

$$\Sigma_{stat} = \Big\{ \ (x,t) \in I\!R^{n+1} \mid x \text{ is a stationary poit of } P(t) \ \Big\}$$

In [8, 9] the local structure of Σ_{gc} is completely described in the neighborhood of generalized critical poits of the following 5 types. (c.f. [3], too)

Type 1:

$(\bar{x}, \bar{t}) \in \Sigma_{gc}$ is of Type 1 if the following conditions are fulfilled:

A1) LICQ is satisfied at $\bar{x} \in M(\bar{t})$.
Then there exits only one $\bar{u}_j$, $j \in I \cup K_0(\bar{x}, \bar{t})$ such that (9) is satisfied with $\bar{u}_0 = 1$

A2) $D_x L(\bar{z}) = 0$, where $\bar{z} = (\bar{x}, \bar{t})$.

A3) $\bar{u}_j \neq 0, \ j \in K_0(\bar{z})$.

A4) $D_x^2 \, L|_{T_{\bar{z}}M}$ is nonsingular.

where $L(z)$ denotes the Lagragian :

$$L(z) = f(z) + \sum_{k \, \in \, I \cup K_0(\bar{z})} \bar{u}_k \, g_k(z)$$

and $T_{\bar{z}}M$ the tangent subspace:

$$T_{\bar{z}}M = \{ \ \xi \in I\!R^n \mid D_x g_k(\bar{z}) \, \xi = 0, \ k \in I \cup K_0(\bar{z}) \ \}$$

$D_x^2 L|_{T_z M}$ represents $V^T D_x^2 L V$, where V is a matrix whose columns form a basis for $T_z M$.

The 2-5 types represent 4 basic degeneracies of Type 1.

Type 2.- The violation of A3.

Type 3.- The violation of A4.

Type 4.- The violation of A1, but $\mid I \mid + \mid K_0(z) \mid < n + 1$.

Type 5.- The violation of A1, but $\mid I \mid + \mid K_0(z) \mid = n + 1$.

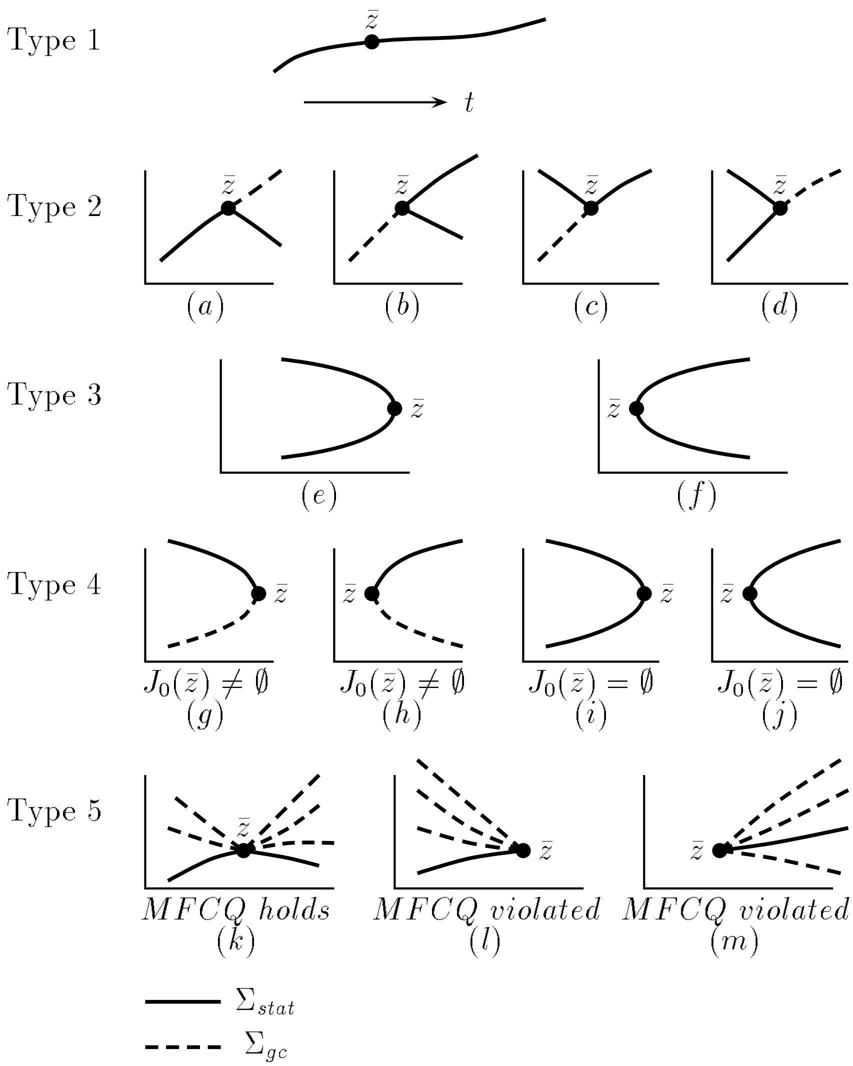

Figure 1: The 5 types of generalized critical points.

$$\mathcal{F} = \left\{ (f,g) \in C^3(I\!R^{n+1}, I\!R^{p+1}) \;\middle|\; \begin{array}{c} \text{each point of } \Sigma_{gc} \\ \text{belongs to one of the 5 types} \end{array} \right\}$$

According to the investigations [8, 9] we have the following posibilities for the local structure of Σ_{gc} (see fig. 1)

The set $\mathcal{F}$ is open and dense with to the strong (or Whitney) C^3_s-topology. In [8] is proved the following result:

Theorem 2.1
If $(f,g) \in \mathcal{F}$ then, the set Σ_{KT} is a 1-dimensional manifold with boundary and

$$z \in \Sigma_{KT} \quad \textit{is a boundary point} \quad \Leftrightarrow \quad K_0(z) \neq \emptyset \quad \textit{and MFCQ fails to hold.}$$

where Σ_{KT} (c.f. [8]) is by definition the clousure of the set of all stationary point of Type 1.

With the results presented (c.f. [8]) we obtained a condition ($\mathcal{F}$) for f and g , such that Σ_{gc} is suitable for application of continuation methods (c.f. e.g. [3]). The continuation methods for the singularities are explained in [3] , it is posible to jump from one connected component in Σ_{gc} to another one, if there is no continuation in Σ_{gc}, but not in any cases. (c.f. [3, 6]). It is not posible to jump in many cases of types 4 and 5.

Furthermore we use another regularity condition for (f,g).

Let $z = (x,u,t) \in I\!R^{n+p+1}$, and $u^+ = \max(u,0)$, $u^- = \min(u,0)$. Define the Kojima map: (c.f. e.g.[11])

$$H(z) = \begin{bmatrix} D_x f(x,t) & + & \sum_{i\in I} u_i D_x g_i(x,t) & + & \sum_{j\in K} u_j^+ D_x g_j(x,t) \\ -g_1(x,t) & & & & \\ \vdots & & \ddots & & \vdots \\ -g_m(x,t) & & & & \\ u_{m+1}^- & - & g_{m+1}(x,t) & & \\ \vdots & & \ddots & & \vdots \\ u_p^- & - & g_p(x,t) & & \end{bmatrix} \tag{10}$$

Lemma 2.1 *(c.f. [10])*
If $0 \in I\!R^{n+p}$ is a regular value of H and MFCQ is satisfied at all $(x,t) \in \Sigma_{stat}$ then, Σ_{stat} is a one-dimensional manifold and each connected component of $H^{-1}(0)$ is homeomorfic with a circle (loop) or with $I\!R$ (path).

Here is H a PC^1 mapping as a generalization of C^1 mappings and analogously regular value.

Definition 2.4
Let $T \subset \mathbb{R}$. We define:

1. *(f, g) (or the problem $P(t)$) is JJT-regular with respect to T ,denoted by $(f, g) \in \mathcal{F}|_T$ (or $P(t) \in \mathcal{F}|_T$), if and only if each point of $\Sigma_{gc} \cap \mathbb{R}^n \times T$ belongs to one of the 5 types.*

2. *$P(t)$ is KH-regular on T if 0 is a regular value of the Kojima map (10) of $P(t)$ restricted to $\mathbb{R}^n \times \mathbb{R}^p \times T$.*

3 Singularities

In these section we analize the singularities for the parametrizations (8). In the general case the 5 types of singularities may appear when we follow curves in Σ_{gc}.

If MFCQ is fulfilled in M (from the original problem (1, 2)),it is known that $M_2(t)$ and $M_3(t)$ satisfy MFCQ for all $t \in [0, 1]$ (c.f. e.g. [3, 4]). Using Lemma 4.1 of [8] we obtain that, following Σ_{KT} in $P_i(t)$, $i = 2, 3$,points of Type 4 may not appear and points of Type 5 only in the "good" case (with continuation) (c.f. e.g. [3]). (see the figures of section 2).

We have examples from all the posible singularities for the 3 parametrizations, . We recall that ,when we follow Σ_{gc}, the previous reduction of Type 4 is not posible, as shown in [3, 4].

We give a completion to the tables of singularities presented in [3, 4] and examples that show the differences.

Sing.	Σ_{KT}	Σgc
$P_1(t)$	1,2,3,4,5	1,2,3,4,5
$P_2(t)$	1,2,3,5	1,2,3,4,5
$P_3(t)$	1,2,3,5	1,2,3,4,5

Example 3.1 *(Parametrization 2) :*

$$\lambda_1^0 = \lambda_2^0 = 1, \quad f_1(x_1, x_2) = -x_1; \quad f_2(x_1, x_2) = x_2$$
$$g_1(x_1, x_2) = x_1^3 - 3x_1 - x_2$$
$$\mu^0 = (-1.5, 0), \quad \mu^1 = (3, -3)$$

For this example the following point is of Type 4.
t=0.345155, $x_1 = 0.816558$, $x_2 = -1.90522$, $v = -0.869755$
$u_1 \to +\infty \quad u_2 \to -\infty \quad u_3 \to +\infty$
Where the u represent the Lagrange multipliers.

Example 3.2 *(Parametrization 3) :*

$$\lambda_1^0 = \lambda_2^0 = 1; \quad f_1(x_1) = x_1^2; \quad f_2(x_1) = x_1$$
$$\mu^0 = (-10, 12); \quad \mu^1 = (10, -12)$$

For this example the following point is of Type 4.
t=0.494318345155, $x_1 = 0.5$, $v = 0.363586$
$u_1 \to +\infty \quad u_2 \to -\infty$
In the two points of Typ 4 presented is posible to jump , because they are not endpoints of a curve of local minimizers. (see [6]). It is not difficult to make an example with one point of Typ 4 without jump in the first parametrization.

We know now that could be imposible find a goal realizer ($t = 1$) using the first parametrization. We need a numerical description of all connected components in Σ_{gc} in the worst case.

4 Regularizations

We begin these section with the introduction of the Kojima regularization in details.(c.f. e.g. [11])

We call a partition of $\mathbb{R}^n$ a countable family Q of polyedrons such that:

1. $\bigcup_{S \in Q} S = \mathbb{R}^n$
2. The intersection of any two elements of Q is empty or is a common face
3. Q is locally finite

Given a partition Q of $\mathbb{R}^n$ in polyedrons, we call a function $H : \mathbb{R}^n \to \mathbb{R}^q$ PC^r differenciable ($H \in PC^r(\mathbb{R}^n, \mathbb{R}^q)$) with respect to Q if:
$\forall\, S \in Q$ there is a neighborhood U_S of S and a function H_S such that

1. $\mathrm{H}_S \in C^r(U_S, \mathbb{R}^q)$
2. $\mathrm{H}_S|_S = H|_S$

$b \in \mathbb{R}^q$ is a regular value of $H \in PC^r(\mathbb{R}^n, \mathbb{R}^q)$ on V (open subset of $\mathbb{R}^n$) with respect to Q (partition of $\mathbb{R}^n$), if $\forall\, S \in Q$ the following conditions hold:

1. $\exists\, (U_S, H_S)$ such that b is a regular value of H_S on $U_S \cap V$.
2. $\forall\, x \in H^{-1} \cap \eta \cap V$, where η is any face of S,

$$\text{range} \begin{bmatrix} DH_S(x) \\ B \end{bmatrix} = q + n - \bar{q} \tag{11}$$

where B is a matrix of size $(n-\bar{q}) \times n$ with range $(n-\bar{q})$ and $d \in \mathbb{R}^{n-\bar{q}}$ such that

$$\text{span}(\eta) = \{\ x \in \mathbb{R}^n \mid B\,x \;=\; d\ \}$$

The equation (11) must hold for any representation (B, d) of span(η). (c.f. e.g. [11])

For H as in (10) we have that $H \in PC^1(\mathbb{R}^{n+p+1}, \mathbb{R}^{n+p})$ with respect to the following partition of $\mathbb{R}^n$.

Polyedrons:

$$P(S) = \{\ z \in \mathbb{R}^{n+p+1} \mid u_j \geq 0,\ j \in S;\ u_k \leq 0,\ \ k \in K \setminus S\ \}$$

where $S \subset K$

Faces :

$$\eta(A,B,C) = \{\, z \in \mathbb{R}^{n+p+1} \mid u_j \geq 0,\ j \in A,\ \ u_r = 0,\ \ r \in B,\ \ u_k \leq 0,\ \ k \in C\,\}$$

where (A, B, C) is a partition of K with $B \neq \emptyset$.

The faces of $P(S)$ have of the form $\eta(A, B, K \setminus (A \cup B))$ with $A \subset S$, $S \subset (A \cup B)$. For a polyedron $P(S)$ we obtain the corresponding reduction of H:

$$H_{P(S)} = \begin{bmatrix} D_x f(x,t) & + & \sum_{i \in I \cup S} u_i D_x g_i(x,t) & \\ -g_i(x,t) & & & i \in I \cup S \\ u_j & - & g_j(x,t) & j \in K \setminus S \end{bmatrix} \tag{12}$$

We need the following important result.

Theorem 4.1 (Sard)
Let V an open subset of $\mathbb{R}^n$

1. *Let $\phi \in C^r(V, \mathbb{R}^q)$ and $r > (n-q)^+$. Then the set of singular values of ϕ has Lebesgue measure 0.(c.f. e.g. [12])*
2. *(Parametrized): Let $\phi \in C^r(\mathbb{R}^{n+p}, \mathbb{R}^q)$ and $r > (n-q)^+$. If $0 \in \mathbb{R}^q$ is a regular value of ϕ over $V \times \mathbb{R}^p$, then for almost all $\alpha \in \mathbb{R}^p$ 0 is a regular value of the function $\phi_\alpha : V \to \mathbb{R}^q$ defined as $\phi_\alpha(x) = \phi(x, \alpha)$. (c.f. e.g. [1])*

The parametrized version remainds true if we consider ϕ defined on $V \times W$, where $V \in \mathbb{R}^n$ and $W \in \mathbb{R}^p$ are open subsets. We will often refer to this last case of Sard's theorem .

For $P_1(t)$ we introduce the following perturbed problem

$$\begin{array}{rlrclcl} \tilde{P}_1(t) & : min & F(x) & + & (1-t)\, c^T x & & \\ restricted & to & G_i(x) & + & (1-t)\, b_i & = & 0,\ i \in I \\ & & G_j(x) & + & (1-t)\, b_j & \leq & 0,\ j \in K \\ & & f_k(x) & - & \mu_k(t) & \leq & 0,\ k \in L \end{array} \tag{13}$$

Where $F(x) = \sum_{k=1}^{L} \lambda_k^0 \, f_k(x)$ and $G = (g_1, \ldots, g_p)$.

Theorem 4.2
Let f, G, λ^0 such that $f, D_x f, G$ and $D_x G$ are twice continously differentiable, then for almost all $(c, b, \mu^0, \mu^1) \in \mathbb{R}^{n+p+2L}$ $\tilde{P}_1(t)$ is KH-regular on $\mathbb{R} \setminus \{1\}$

<u>Proof:</u>
Let us introduce the following notation:

$$\begin{array}{rcl} \tilde{L} & = & p+1, \ldots, p+L \\ \tilde{K} & = & K \cup \tilde{L} \\ g_{k+p}(x,t) & = & f_k(x) - \mu_k(t), \quad k \in L \end{array}$$

We call $\tilde{H}$ the Kojima function (10) of the problem (13). For $S \subset \tilde{K}$ the following relation is fulfilled

$$\tilde{H}_S = H_S + (1-t) \begin{bmatrix} c^T \\ -b^T \\ 0 \end{bmatrix}$$

Computing the derivative with respect to the variables c, b, μ^0 and μ^1 we obtain that 0 is a regular value of $\tilde{H}_S$ restricted to the set $\mathbb{R}^{n+p+1}(t \neq 1)$ where

$$\mathbb{R}^{n+p+1}(t \neq 1) = \{ z \in \mathbb{R}^{n+p+1} \mid \quad t \neq 1\} \tag{14}$$

From Theorem 4.1 we obtain that for almost all (c, b, μ^0, μ^1) $\tilde{H}_S$ restricted to $\mathbb{R}^{n+p+1}(t \neq 1)$ has 0 as a regular value.

Now, for the second condition of (11). Let η a face of $P(S)$ for $S \subset K$, then there is an index set B with $B \subset K$ such that.

$$\mathrm{span}(\eta) = \{ \ z \in \mathbb{R}^{n+p+1} \mid \quad M_B \ z = 0 \ \}$$

Where M_B is an $(n+p+1-\dim(span(\eta))) \times (n+p+1)$ matrix whose columns having index in B form an identity square matrix and all other elements are 0.

It is easy to verify that over the set $\mathbb{R}^{n+p+1}(t \neq 1) \times \mathbb{R}^{n+p+2L}$ the matrix

$$\begin{bmatrix} DH \\ M_B \end{bmatrix} \tag{15}$$

has full range.We note that if we change the identity of M_B for a regular matrix the matrix (15) has also full range.

We obtain using theorem 4.1 that for almost all (c, b, μ^0, μ^1) the condition (11) for η is fulfilled.

The numbers of faces of $P(S)$ is finite and so is the number of polyedrons, we obtain then that for almost all (c, b, μ^0, μ^1) 0 is a regular value of $\tilde{H}$ restricted to $\mathbb{R}^{n+p+1}(t \neq 1)$. Also $\tilde{P}(t)$ is regular in the sense of Kojima and Hirabayashi. $\square$

For the other parametrizations we obtain, similarly the following result

Theorem 4.3
Let f, G, λ^0 such that $f, D_x f, G$ and $D_x G$ are twice continously differenciable, then for almost all $(c, b, \mu^0, \mu^1) \in \mathbb{R}^{n+p+2L}$ $\tilde{P}_2(t)$ and $\tilde{P}_3(t)$ is KH-regular on $\mathbb{R}\{1\}$ in the sense of Kojima, except for the points where $t = 1$.

$\tilde{P}_2(t)$ and $\tilde{P}_3(t)$ are defined anagously , with an aditional parameter c_{n+1}.

For the regularization in the sense of Jongen, Jonker and Twilt we consider the following perturbed problem:

$$\begin{array}{rcl} \bar{P}_1(t) & : min & F(x) + (1-t)\ (\frac{1}{2}\ x^T A x + c^T x) \\ restricted to & : & x \in \bar{M}(t, B, x^0, \mu^0, \mu^1) \end{array} \tag{16}$$

where

$$\begin{array}{rcl}\bar{M}(t,B,x^0,\mu^0,\mu^1) &=& \{\ x\in I\!\!R^n \mid \bar{g}_i(x,t)=0,\ i\in I,\ \bar{g}_j(x,t)\le 0,\ j\in\tilde{K}\ \} \\ \bar{g}_i(x,t) &=& t\ g_i(x,t)+(1-t)\ b_i^T\ (x-x^0),\quad i\in I \\ \bar{g}_j(x,t) &=& t\ g_j(x,t)+(1-t)\ (b_i^T\ x+d_{j-m}),\quad j\in K \\ \bar{g}_k(x,t) &=& t\ f_{k-p}(x,t)+(1-t)\ b_i^T x-\mu_{k-p}(t),\quad k\in\tilde{K}\end{array}$$

Here $A\in I\!\!R^{0{,}5n(n+1)}$ represents a symetric matrix, $B\in I\!\!R^{n(p+L)+p-m}$ with $B=(b_1,\ldots,b_{p+L},d)$ where $b_j\in I\!\!R^n$, $j\in I\cup\tilde{K}$, $d\in I\!\!R^{p-m}$ and $x^0\in I\!\!R^n$.

Theorem 4.4
If the 3rd partial derivatives of f_k, $k\in L$ and g_j, $j\in I\cup K$ are continuosly differentiable with respect to x, then for almost all (A,c,B,x^0,μ^0,μ^1) follows $\bar{P}_1(t)\in\mathcal{F}|_{I\!\!R\setminus\{1\}}$

We use "almost all" in the following sense: each measurable subset of the set

$$\left\{(A,c,B,x^0,\mu^0,\mu^1)\in I\!\!R^{0.5n(n+1)+n(p+L+2)+p-m+2L}\,|\,\bar{P}_1(t)\notin\mathcal{F}|_{I\!\!R\setminus\{1\}}\right\}$$

has Lebesgue measure zero.

We use the following notation for a given (B,x^0,μ^0,μ^1)

$$\begin{array}{rcl}\bar{K}_0(x,t) &=& \{\ j\in\tilde{K}\mid \bar{g}_j=0\ \} \\ \bar{M}_F(B,x^0,\mu^0,\mu^1) &=& \{(x,t)\in I\!\!R^{n+1}|\ x\in\bar{M}(t,B,x^0,\mu^0,\mu^1),\ \text{LICQ fails}\}\end{array}$$

Lemma 4.1
For almost all (B,x^0,μ^0,μ^1) we have that

1. *$\bar{M}_F(B,x^0,\mu^0,\mu^1)\ \cap\ \{\ (x,t)\in I\!\!R^{n+1}\ \mid\ t\neq 1\ \}$ is a 0-dimensional manifold.*
2. *If $(\bar{x},\bar{t})\in\bar{M}_F(B,x^0,\mu^0,\mu^1)\cap\{(x,t)\in I\!\!R^{n+1}|\ t\neq 1\}$, then the vectors $\{Dg_j(\bar{x},\bar{t}),\ j\in I\cup\bar{K}_0(\bar{x},\bar{t})\}$ are linearly independent and if λ_j, $j\in I\cup\bar{K}_0(\bar{x},\bar{t})$ are solution of*

$$\sum_{j\,\in\,I\cup\bar{K}_0(\bar{x},\bar{t})}\lambda_j\ D_x g_j(\bar{x},\bar{t})=0$$

then $\lambda_j\neq 0$, for $j\in\bar{K}_0(\bar{x},\bar{t})$

Proof of Lemma 4.1:
Using the Theorem of Fubini (c.f. e.g. [7]) and an inductive argument we obtain that the complement of the open set

$$B' = \left\{ (B, x^0, \mu^0, \mu^1) \in I\!R^{n(p+L)+p-m} \mid b_i,\ i \in I \text{ are linearl. indep.} \right\} \quad (17)$$

has Lebesgue measure 0.

Let $S \subset \tilde{K}$, $q \in I \cup S$ and $\tilde{S} \subset S$ be fixed. We consider the functions

$$\begin{aligned}
\Gamma(x, \kappa, t, B, x^0, \mu^0, \mu^1) &= \sum_{j \in (I \cup S) \setminus \{q\}} \kappa_j \, \bar{g}_j(x,t) + \bar{g}_q(x,t) \\
\Upsilon_1(x, \kappa, t, B, x^0, \mu^0, \mu^1) &= \begin{bmatrix} D_x^T \Gamma(x, \kappa, t, B, x^0, \mu^0, \mu^1) \\ \bar{g}_j(x,t), \quad j \in I \cup S \end{bmatrix} \\
\Upsilon_2(x, \kappa, t, B, x^0, \mu^0, \mu^1) &= \begin{bmatrix} \Upsilon_1(x, \kappa, t, B, x^0, \mu^0, \mu^1) \\ \kappa_j \quad j \in \tilde{S} \end{bmatrix}
\end{aligned}$$

Where κ is formed by the κ_j, $j \in S$

Computing the derivatives of Υ_1 with respect to b_q, d_j, $j \in S \cap K$ x^0 and μ^0_k, $p+k \in S$ and Υ_2 with respect to the same variables and with respect to κ_j, $j \in \tilde{S}$ too we obtain that Υ_1 and Υ_2 restricted over the set

$$I\!R^{n+|S|+1}(t \neq 1) \times B'$$

where $I\!R^{n+|S|+1}(t \neq 1)$ is defined anagously to (14), have 0 as a regular value.

Using the theorem 4.1 and that B' has complement with Lebesgue measure 0 we obtain ,that for almost all (B, x^0, μ^0, μ^1), Υ_1 and Υ_2 have 0 as a regular value over the set $I\!R^{n+|S|+1}(t \neq 1)$.

Since the ways to select S and $\tilde{S}$ are finite we obtain that for almost all (B, x^0, μ^0, μ^1), Υ_1 and Υ_2 have 0 as a regular value over the set $I\!R^{n+s+1}(t \neq 1)$ for each selection of S, $\tilde{S}$ and q.

The Assertions of the Lemma can be easily obtained using the Jacobi matrix of Υ_1 and Υ_2. □

Proof of Theorem 4.4:
We make the proof in two steps:

Step 1:
We prove here that for almost all $(A, c, B, x^0, \mu^0, \mu^1)$,each generalized critical point with LICQ and $t \neq 1$ is of Type 1, 2 or 3.

Let $S \subset \tilde{K}$ fixed. Supose that $S = \{m+1, \ldots, q\}$ with $q \geq m$. Let $T \subset \{1, \ldots, n+q\}$ and $\bar{T} \subset S$.

Let $M^{n+q}(T) \subset I\!R^{0,5(n+q)(n+q+1)}$ the set of all symmetric matrix with size $(n+q) \times (n+q)$ and range $|T|$, such that the columns with index in T are linearly independent.

$M^{n+q}(T)$ is a manifold of codimension $\frac{1}{2}(n+q-|T|)(n+q-|T|+1)$. We denote by the equations that define these manifold locally as ξ

We use H and H_S as the Kojima matrix (10) and the corresponding reduction (12) for a problem such that the following relation is fulfilled:

$$\bar{H}_S = H_S + \begin{bmatrix} (1-t)\ [\ Ax + c + \sum_{i \in I \cup S} u_i\ b_i\ ] \\ -(1-t)\ b_i^T\ (x - x^0) \quad i \in I \\ -(1-t)\ [\ b_j^T\ x + d_j\ ] \quad j \in K \\ -(1-t)\ b_k^T\ x \quad k \in \tilde{L} \end{bmatrix} \tag{18}$$

where $\bar{H}_S$ the corresponding reduction (12) of the Kojima matrix (10) of the problem (16).

If we derivate the equation (18) with respect to x and u_i, $i \in I \cup S$ the rows with index l such that $l = 1, \ldots, n$ or $(l-n) \in I \cup S$, we obtain (by definition)

$$\bar{M}_S = M_S + \begin{bmatrix} (1-t)\, A & (1-t)\, b_j \quad j \in I \cup S \\ -(1-t)\, b_j^T \quad j \in I \cup S & 0 \end{bmatrix}$$

The size of $\bar{M}_S$ and M_S are $(n+q) \times (n+q)$, but they are not symmetric. We consider also the matrix

$$\bar{M}_S^+ = \bar{M}_S \quad \begin{bmatrix} -I_n & 0 \\ 0 & I_q \end{bmatrix}$$

which is symmetric and has the same rank as $\bar{M}_S$.

We will consider all the symmetric matrices of size $l \times l$ as elements of the space $I\!R^{0,5(l+1)l}$.

We define the function Ω defined over

$$I\!R^{2(n+p+L)+1+0,5(n+q)(n+q+1)+0,5n(n+1)+2n+n(p+L)+p-m+2L}$$

and with values on

$$I\!R^{2(n+p+L)+0,5(n+q)(n+q+1)+0,5(n+q-|T|)(n+q-|T|+1)+|\bar{T})|}$$

We consider the variable of Ω ,that we call s, subdidived as

$$s = (z, \bar{z}, \tilde{z}, A, c, B, x^0, \mu^0, \mu^1)$$

Where $z = (x, u, t) \in I\!R^{n+p+L+1}$, $\bar{z} \in I\!R^{n+p+L}$ and $\tilde{z} \in I\!R^{0,5(n+q)(n+q+1)}$.

We denote by e_j the canonical vectors .

$$\Omega(s) := -\bar{z}_j + e_j \ \bar{H}_S(z, A, c, B, \mu^0, \mu^1)$$

for $j = 1, \ldots, n+p+L$

$$\Omega_{n+p+L+j}(s) := -\tilde{z}_j + e_j \ \bar{M}_S^+(z, A, c, B, \mu^0, \mu^1)$$

for $j = 1, \ldots, 0,5(n+q)(n+q+1)$

$$\Omega_{n+p+L+0,5(n+q)(n+q+1)+j}(s) := \bar{z}_j$$

for $\quad j = 1, \ldots, n+p+L$

$$\Omega_{2(n+p+L)+0,5(n+q)(n+q+1)+j}(s) := \xi_j(\tilde{z})$$

for $j = 1, \ldots, 0,5(n+q-|T|)(n+q-|T|+1)$

$$\Omega_{2(n+p+L)+0,5(n+q)(n+q+1)+0,5(n+q-|T|)(n+q-|T|+1)+j}(s) := u_j$$

for $j = 1, \ldots, |\bar{T}|$.

It can be proved that if $\{n+1, \ldots, n+q\} \subset T$ then 0 is a regular value of Ω defined over the open set

$$\{ \ (z, \bar{z}, \tilde{z}, A, c, B, x^0, \mu^0, \mu^1) \mid t \neq 1, \ (B, x^0, \mu^0, \mu^1) \in B' \ \}$$

where B' is defined by (17).(see [2] ,for details) With the same idea used in the lema 4.1 can be obtained that for almost all $(A, c, B, x^0, \mu^0, \mu^1)$ and for each selection of S, T, $\bar{T}$ such that

$$\{n+1, \ldots, n+q\} \subset T$$

, that means LICQ, Ω has 0 as regular value over the open set

$$Z(t \neq 1) = \{ \, (z, \bar{z}, \tilde{z}) \, | \, t \neq 1 \, \}$$

Supose that we have a "good" selection of $(A, c, B, x^0, \mu^0, \mu^1)$, and we supose that these selection is 0 , we obtain $\bar{H} = H$.

For fixed T and $\bar{T}$, Ω is defined over

$$\mathbb{R}^{2(n+p+L)+1+0,5(n+q)(n+q+1)}$$

and with values on

$$\mathbb{R}^{2(n+p+L)+0,5(n+q)(n+q+1)+0,5(n+q-|T|)(n+q-|T|+1)+|\bar{T})|}$$

Since 0 is a regular value of Ω, we have then only 3 posibilities:

1. $0,5(n+q-|T|)(n+q-|T|+1)+|\bar{T})| > 1$, and then $\Omega^{-1}(0) \cap Z(t \neq 1)$ is empty.
2. $0,5(n+q-|T|)(n+q-|T|+1)+|\bar{T})| = 1$, and then $\Omega^{-1}(0) \cap Z(t \neq 1)$ is a 0-dimensional manifold.
3. $0,5(n+q-|T|)(n+q-|T|+1)+|\bar{T})| = 0$, and then $\Omega^{-1}(0) \cap Z(t \neq 1)$ is a 1-dimensional manifold.

If $(z, \bar{z}, \tilde{z}) \in \Omega^{-1}(0) \cap Z(t \neq 1)$ then there is only 3 posibilities:

Case 1 $|T| = n+q$ and $\bar{T} = \emptyset$

Case 2 $|T| = n+q-1$ and $\bar{T} = \emptyset$

Case 3 $|T| = n+q$ and $|\bar{T}| = 1$

For each point $z = (x,t) \in \Sigma_{gc}$ where LICQ is satisfied we obtain that there exits S, T and $\bar{T}$ such that:

$$(z, H_S(z), M_S^+(z)) \in \Omega^{-1}(0)$$

Choosing

$$S = \tilde{K}_0\ (x,t)$$
$$\bar{T} = \{\ j \in \tilde{K}_0(x,t) \mid u_j = 0\ \}$$

and T an index set such that $|T| = \text{range}(\ M_S(z)\ )$, the columns of $M_S(z)$ with index in T linearly independent and

$$\{\ n+1, \ldots, n+m+\ |\tilde{K}_0(x,t)|\ \} \quad \subset \quad T$$

The 3 cases (Case 1, 2 and 3) represents the Typ 1, 3 and 2. (c.f. [11])

Step 2:

Taking into acount the two following facts:

1. Let $(\bar{x}, \bar{t})$ fixed with $(t \neq 1)$, then for almost all $(A, c) \in I\!R^{0,5n(n+1)+n}$ it holds that:

$$t\, D_x f(\bar{x}, \bar{t}) + (1-t)\,[A\,\bar{x}\; + \; c] \notin \text{ span} \,\{\, D_x g_i(\bar{x}, \bar{t}),\; i \in I \cup S \,\}$$

   for each $S \subset \tilde{K}$ with $\{\, D_x g_i(\bar{x}, \bar{t}),\; i \in I \cup S\} \;<\; n$.

2. If $I\!R$ is a matrix with range greater than 0 and $(t \neq 1)$ then for almost all $(A, c) \in I\!R^{0,5n(n+1)+n}$ it holds that:

$$[\, t\, D_x f(\bar{x}, \bar{t}) + (1-t)\, A\,\bar{x}\; + \; c]^T \quad I\!R \quad [\, t\, D_x f(\bar{x}, \bar{t}) + (1-t)\, A\,\bar{x}\; + \; c] \neq 0$$

and using the same argument as in [11] it can be proved that if $(B, x^0, \mu^0, \mu^1) \in B'$, B' defined by (17) then for almost all $(A, c) \in I\!R^{0,5n(n+1)+n}$ each generalized critical point without LICQ is of Typ 4 or 5.

We note that we can get the parameters so small as needed, and such that, the perturbed problem (16) belongs to the set $\mathcal{F}(t \neq 1)$ and x^0 is a feasible point for $\bar{P}_1(0)$. The same property is not posible to hold if we want that x^0 be a generalized critical point too.

For $P_2(t)$ and $P_3(t)$ we obtained similar regularizations results (see [2]).

We are now interested in the MFCQ condition for the set $M_1(t)$.

Theorem 4.5
Let $\bar{L} \subset L$ not empty. If $\bar{x}$ is a locally efficient points, a ϵ-locally efficient point or a weakly locally efficient point of the problem:

$$min \quad \{\; f_j(x),\;\; j \in \bar{L} \mid x \in M \;\} \tag{19}$$

then MFCQ is not fulfilled globally over the set

$$\tilde{M} = \{\; x \in I\!R^n \mid x \in M,\;\; f_k(x) \leq f_k(\bar{x}),\;\; k \in \bar{L} \;\}$$

Proof:
Supose $\bar{L} = \{1, \ldots, l\}$ with $l \leq L$. If MFCQ fails to hold in $\bar{x}$ for M then fails to hold for $\tilde{M}$ too. We can then supose that MFCQ is fulfilled for these case.

If MFCQ is fulfilled at $\bar{x}$ for M, then there exists a function $x(t)$ such that:

- $x(0) = \bar{x}$

- $x(t) \in \tilde{M} \subset M \quad \forall\, t \in (0, \bar{t})$

We obtain using a continuity argument of f_k, $k \in \bar{L}$ that there exists a positive $\tilde{t} \leq \bar{t}$ such that:

$$f_k(x(t)) < f_k(\bar{x}) \quad \forall\, t \in (0, \tilde{t}), \quad \forall\, k \in \bar{L}$$

That is a contradiction with the asumption that $\bar{x}$ is a locally efficient point, a ϵ-locally efficient point or a weakly locally efficient point of the problem (19) $\square$

We note that if $\bar{x}$ is a point of interest , then there exists a subset of $I\!R^L$ which can't be intersected by the segment $\mu(t)$, $t \in [0,1]$, if we are interested in MFCQ for $M_1(t)$, $t \in [0,1]$.

The next theorem give us a measure of these "bad" set in $I\!R^{2L}$.

Theorem 4.6
Let $k \in L$, such that the set

$$M(\mu_k) = \{\ x \in I\!R^n \mid x \in M, \quad f_k(x) \leq \mu_k\ \}$$

doesn't fulfill MFCQ for some $\bar{\mu}_k \in I\!R$. Then the set

$$\{\ (\mu^0, \mu^1) \in I\!R^{2L} \mid \exists t \in [0,1] \text{ where MFCQ fails to hold for } M_1(t)\ \} \quad (20)$$

has a subset of measure $+\infty$.

<u>Proof:</u>
Supose that $k = 1$, and at $\bar{x} \in M(\bar{\mu}_1)$ MFCQ fails to hold. We consider the following subset of $I\!R^L$

$$\tilde{I\!R}^L = \{\ \mu \in I\!R^L \mid \mu_1 = \bar{\mu}_1, \quad \mu_j \geq f_j(\bar{x}),\ j = 2, \ldots, L\ \}$$

and using $\tilde{I\!R}^L$ we define

$$\tilde{I\!R}^{2L} = \{\ (\mu^0, \mu^1) \in I\!R^{2L} \mid \exists\ t \in [0,1] \text{ with } (1-t)\,\mu^0 + t\,\mu^1 \in \tilde{I\!R}^L\ \}$$

We will prove that the set $\tilde{I\!R}^{2L}$, a subset of (20) , has L-measure $+\infty$.

It's easy to prove that $\tilde{I\!R}^{2L}$ is closed and then measurable.

We consider the variable $(\mu^0, \mu^1) \in I\!R^{2L}$ subdivided in the following way $(\tilde{\mu}^0, \tilde{\mu}^1) \in I\!R^{2L}$ where:

$$\tilde{\mu}^0 = (\mu^0, \mu_1^1) \in I\!R^{L+1} \quad \text{and} \quad \tilde{\mu}^1 = (\mu_2^1, \ldots, \mu_L^1) \in I\!R^{L-1}$$

If $\tilde{\mu}^0 \notin I\!R_0 \subset I\!R^{L+1}$ then the set $I\!R_1(\tilde{\mu}^0) \subset I\!R^{L-1}$ is empty, where

$$I\!R_0 = \{ \ \tilde{\mu}^0 \in I\!R^{L+1} \mid \mu_1^0 \leq \bar{\mu}_1 \leq \mu_1^1 \text{ or } \mu_1^0 \geq \bar{\mu}_1 \geq \mu_1^1 \ \}$$

and

$$I\!R_1(\tilde{\mu}^0) = \{ \ \tilde{\mu}^1 \in I\!R^{L-1} \mid (\tilde{\mu}^0, \tilde{\mu}^1) \notin \tilde{I\!R}^{2L} \ \}$$

If $\tilde{\mu}^0 \in I\!R_0$ is such that $\mu_1^0 \neq \bar{\mu}_1$ then , defining $\bar{t}$ as

$$\bar{t} = \frac{\bar{\mu}_1 - \mu_1^0}{\mu_1^1 - \mu_1^0}$$

we obtain the representation

$$I\!R_1(\tilde{\mu}^0) = \{ \ \tilde{\mu}^1 \in I\!R^{L-1} \mid \mu_j^1 \geq \frac{1}{\bar{t}} \ [\, f_j(\bar{x}) + (\bar{t} - 1)\, \mu_j^0 \,] \ , \ j = 2, \ldots, L \ \}$$

This set has measure $+\infty$ in $I\!R^{L-1}$. Using that the set $I\!R_0$ has L-measure $+\infty$ in $I\!R^{L+1}$ and the theorem of fubini we obtain the result. $\Box$

Remark:

If the original set M of (2) has a connected compact component, then the set (20) has a subset of measure $+\infty$.

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